MATH 2040A Lecture 1 (Sep 8, 2016)

<u>Revision</u> (Textbook Ch. 1-4)

$\boxed{\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}}$ in this course.

<u>Def$^{\underline{n}}$</u>: A vector space over $\mathbb{F}$ consists of

$$(V, +, \cdot) \quad \text{s.t.} \quad \begin{cases} \vec{u}+\vec{v} \in V \\ a\cdot\vec{u} \in V \end{cases}$$

V: set of vectors; $+$: vector addition; $\cdot$: scalar multiplication

which satisfies 8 properties:

(VS1) $\vec{x}+\vec{y} = \vec{y}+\vec{x} \quad \forall \vec{x},\vec{y} \in V$

(VS2) $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z}) \quad \forall \vec{x},\vec{y},\vec{z} \in V$

(VS3) $\exists \vec{0} \in V$ s.t. $\vec{x}+\vec{0} = \vec{x} \quad \forall \vec{x} \in V$

(VS4) $\forall \vec{x} \in V, \exists \vec{y} \in V$ s.t. $\vec{x}+\vec{y} = \vec{0}$

(VS1)–(VS4): $+$

(VS5) $1\cdot\vec{x} = \vec{x} \quad \forall \vec{x} \in V$

(VS6) $(ab)\cdot\vec{x} = a\cdot(b\cdot\vec{x}) \quad \forall a,b \in \mathbb{F}, \forall \vec{x} \in V$

(VS5)–(VS6): $\cdot$

(VS7) $a\cdot(\vec{x}+\vec{y}) = a\cdot\vec{x} + a\cdot\vec{y} \quad \forall a \in \mathbb{F}, \forall \vec{x},\vec{y} \in V$

(VS8) $(a+b)\cdot\vec{x} = a\cdot\vec{x} + b\cdot\vec{x} \quad \forall a,b \in \mathbb{F}, \forall \vec{x} \in V$

(VS7)–(VS8): $+$ & $\cdot$

Examples of vector spaces:

(a) $\mathbb{R}^n$, $\mathbb{C}^n$

(b) $M_{m\times n}(\mathbb{R}) = \{ m\times n \text{ matrices } / \mathbb{R} \}$.

(c) $P(\mathbb{R}) = \{ \text{polynomials with real coeff.} \}$

$\quad\;\; \cup\!|$

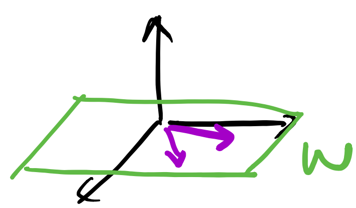
$\quad P_n(\mathbb{R}) = \{ \text{poly. of degree} \le n \}$

(d) $C(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} \text{ continuous} \}$

Subspace: $W \subseteq V$ is a vector space itself

(using $+, \cdot$ as V)

$V \ (= \mathbb{R}^3)$



Prop: $W \subseteq V$ is a subspace iff

(1) $\vec{0} \in W$

(2) $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} + \vec{v} \in W$

(3) $a \in \mathbb{F}, \vec{u} \in W \Rightarrow a \cdot \vec{u} \in W$

Examples of subspace

(a) $\{\vec{0}\}$, V trivial subspaces

(b) $P_n(\mathbb{R}) \subseteq P(\mathbb{R})$

(c) $\{\text{symmetric matrix}\} \subseteq M_{n\times n}(\mathbb{R})$
$A^T = A$

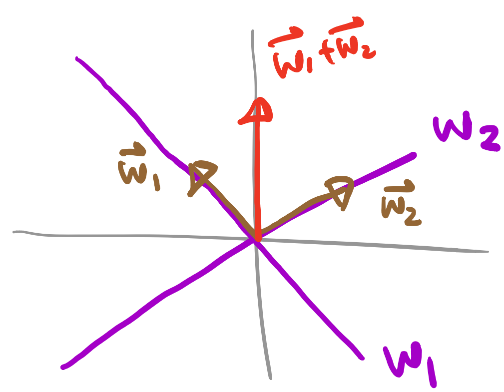
Thm: If $W_1, W_2 \subseteq V$ are subspaces.

then (1) $W_1 \cap W_2 \subseteq V$ subspace

(2) Note: $W_1 \cup W_2$ NOT subspace. (Ex.)

$$W_1 + W_2 := \{\vec{w_1} + \vec{w_2} \mid \vec{w_1} \in W_1, \vec{w_2} \in W_2\}$$

Eg. $V = \mathbb{R}^2$



Span: $S \subseteq V$ subset

The span of S is

$$\text{Span}(S) = \{ \underbrace{a_1\vec{v}_1 + \cdots + a_k\vec{v}_k}_{\text{linear combination of } \vec{v}_1, \ldots, \vec{v}_k} \mid a_i \in \mathbb{F}, \vec{v}_i \in S \}$$

Note: This is the smallest subspace containing S.

Q: If $S \subseteq V$ subspace, then span $S = S$.

Def^n: $S \subseteq V$ is linearly dependent

if $\exists$ a "non-trivial" linear combination

$$\underset{\text{vectors in } S}{a_1\vec{v}_1 + \cdots + a_k\vec{v}_k} = \vec{0}$$

$a_i \neq 0$ for some i.

If not, S is linearly independent.

Thm: $S \subseteq V$ linearly indep

$\Longleftrightarrow$ "If $a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = \vec{0}$, then $a_i = 0$ for all i"

Prop: Suppose $S_1 \subseteq S_2 \subseteq V$ subsets.

Then (1) S_1 lin. dep. $\Rightarrow$ S_2 lin. dep.

(2) S_2 lin. indep. $\Rightarrow$ S_1 lin. indep.

Prop: If $S \subseteq V$ lin. indep, then

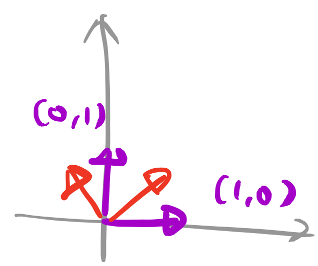
$$S \cup \{\vec{v}\} \text{ lin. dep.} \iff \vec{v} \in \operatorname{Span}(S)$$

Def: $\beta \subseteq V$ is a basis

if (1) $\operatorname{span}(\beta) = V$

(2) β lin. indep.

$V = \mathbb{R}^2$



define $\dim V$ = # of vectors in ANY basis β. $\begin{pmatrix} < +\infty \\ = +\infty \end{pmatrix}$

E.g. of basis

(a) $\mathbb{R}^n$, $\mathbb{C}^n$ $\beta = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ std. basis

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\dim \mathbb{R}^n = n$

(b) $P_n(\mathbb{R})$, $\beta = \{1, x, x^2, \dots, x^n\}$ std basis

$P_n(\mathbb{R}) = \{a_0 + a_1 x + \cdots + a_n x^n\}$

$\dim P_n(\mathbb{R}) = n+1.$